Carathéodory's Theorem

Theorem 1 (Carathéodory)**.** *Let $X \subset \mathbb{R}^d$. Then each point of* $\operatorname{conv}(X)$ *is a convex combination of at most $d+1$ points of X.*

Proof. Suppose there exists $y \in \operatorname{conv}(X)$ that cannot be expressed as a convex combination of fewer than $m \geq d+2$ points in X. Then

$$y = \sum_{k=1}^{m} \lambda_k x_k \text{ with } \sum_{k=1}^{m} \lambda_k = 1 \text{ and } \lambda_k > 0 \, \forall \, k$$

The $m \geq d+2$ points $x_1, \ldots, x_m \in X$ must be affinely dependent, so

$$\sum_{k=1}^{m} \mu_k x_k = 0 \text{ with } \sum_{k=1}^{m} \mu_k = 0$$

Then, for any $\alpha \in \mathbb{R}$,

$$y = y + 0 = \sum_{k=1}^{m} \lambda_k x_k + \alpha \sum_{j=1}^{m} \mu_k x_k = \sum_{k=1}^{m} (\lambda_k + \alpha \mu_k) x_k$$

The new coefficients $\Lambda_k \equiv \lambda_k + \alpha \mu_k$ satisfy $\sum_{k=1}^{m} \Lambda_k = 1$. Choosing

$$j = \arg \min_{k: \mu_k > 0} \frac{\lambda_k}{\mu_k}$$

we further have $\Lambda_k \geq 0$ for all $k = 1, \ldots, m$ and $\Lambda_j = 0$. Hence y is a convex combination of fewer than m points of X, a contradiction! □

From the proof it is clear that each point of $\operatorname{conv}(X)$ for $X \subset \mathbb{R}^d$ can be written as a convex combination of affinely independent points from X, of which there can be at most $d+1$. It follows immediately that the convex hull of a set $X \subset \mathbb{R}^d$ is the union of all simplexes with vertices in X.

Corollary 1. *Let $X \subset \mathbb{R}^d$. Each boundary point of* $\operatorname{conv}(X)$ *is a convex combination of d points from X.*

Proof (from `math.stackexchange.com/q/1786544`). Let $C = \operatorname{conv}(X)$. For any $x \in \partial C$, there is a supporting hyperplane $\mathcal{H}$ to C at x; that is, C is disjoint from an open half-space of $\mathcal{H}$. Observe that any representation of x as a convex combination of points from P cannot involve elements of P that are not in $\mathcal{H}$; otherwise, the combination would lie outside the hyperplane. Therefore $x \in \operatorname{conv}(P \cap \mathcal{H})$. Applying Carathéodory's theorem to $P \cap \mathcal{H}$, considered as a subset of the $(d-1)$-dimensional space $\mathcal{H}$, we are done. □

Corollary 2. *The convex hull of a compact set $K \subset \mathbb{R}^d$ is compact.*

Proof (Danzer et al. 1963). Note that the unit simplex $\Delta^d \subset \mathbb{R}^{d+1}$ is compact. Consider the function $f : (\mathbb{R}^{d+1} \times K^{d+1}) \to K$ given by

$$f(\alpha_1, \ldots, \alpha_{d+1}, x_1, \ldots, x_{d+1}) = \sum_{k=1}^{d+1} \alpha_k x_k \in K$$

Since f is continuous and $\Delta^d \times K^{d+1}$ is compact, the set $f(\Delta^d \times K^{d+1})$ is compact. By Carathéodory's theorem, $f(\Delta^d \times K^{d+1}) = \operatorname{conv}(K)$. □

Radon's Lemma

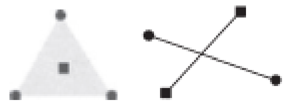

Figure 1: Two radon partitions.

Theorem 2 (Radon's Lemma)**.** *Let $A = \{a_1, \ldots, a_{d+2}\} \subset \mathbb{R}^d$. Then there exist two disjoint subsets $A_1, A_2 \subset A$ whose convex hulls have nonempty intersection.*

Proof (Matoušek 2002). The $d+2$ points in $A \subset \mathbb{R}^d$ must be affinely dependent, that is, there exist $\lambda_1, \ldots, \lambda_{d+2} \in \mathbb{R}$ not all zero such that

$$\sum_{k=1}^{d+2} \lambda_k = 0 \text{ and } \sum_{k=1}^{d+2} \lambda_k a_k = 0$$

The sets $P = \{k \mid \lambda_k > 0\}$ and $N = \{k \mid \lambda_k < 0\}$ determine the desired subsets. Both are nonempty, so put $A_1 = \{\lambda_k \mid k \in P\}$ and $A_2 = \{\lambda_k \mid k \in N\}$. Let $S \equiv \sum_{k \in P} \lambda_k$; we also have $S = -\sum_{k \in N} \lambda_k$. Define

$$x \equiv \sum_{k \in P} \frac{\lambda_k}{S} a_k = \sum_{k \in N} \frac{-\lambda_k}{S} a_k$$

where equality holds because $\sum_{k=1}^{d+2} \lambda_k a_k = \sum_{k \in P} \lambda_k a_k + \sum_{k \in N} \lambda_k a_k = 0$. Both representations cast x as a convex combination, first of points from A_1 then from A_2. Hence $x \in \operatorname{conv}(A_1) \cap \operatorname{conv}(A_2)$. □

Helly's Theorem

Theorem 3 (Helly)**.** *Let $C_1, C_2, \ldots, C_n \subset \mathbb{R}^d$ be convex, with $n \geq d+1$. If every $d+1$ of these sets intersect, then $\cap_{i=1}^n C_i \neq \emptyset$.*

Proof (Matoušek 2002). For fixed d, we proceed by induction on n. The base case $n = d+1$ is clear, so assume $n \geq d+2$ and that Helly's theorem holds for smaller n.

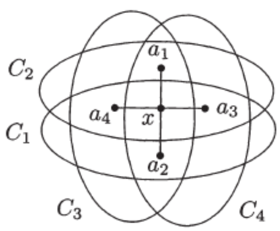

Figure 2: Illustration of Radon's proof of Helly's theorem for $d = 2$ and $n = 4$.

Consider convex $C_1, \ldots, C_n \subset \mathbb{R}^d$ such that any $d+1$ sets intersect. If we leave out any one of these sets C_i, the remaining sets have nonempty intersection $a_i \in \cap_{j \neq i} C_j$ by the inductive assumption. Consider the $n \geq d+2$ points $A = \{a_1, \ldots, a_n\} \subset \mathbb{R}^d$. By Radon's lemma, there exist disjoint sets $A_1, A_2 \subset A$ such that $\operatorname{conv}(A_1) \cap \operatorname{conv}(A_2) \neq \emptyset$. Choose a point x in the intersection. For any $i \in [n]$, either $a_i \notin A_1$ or $a_i \notin A_2$. In the former case, each $a_j \in A_1$ lies in C_i, so $x \in \operatorname{conv}(A_1) \subset C_i$ by convexity. In the latter case we similarly have $x \in \operatorname{conv}(A_2) \subset C_i$. Therefore, $x \in \bigcap_{i=1}^n C_i$. □

Further Reading

Compare proofs to (Matoušek 2002). For a comprehensive survey of applications see (Danzer et al. 1963). An elegant proof of Haar's theorem from approximation theory is given by (Pták 1958) via Carathéodory's theorem.

References

[1] Ludwig Danzer, Branko Grünbaum, and Victor Klee. *Helly's Theorem and its Relatives.* American Mathematical Society Providence, RI, 1963.

[2] Jiří Matoušek. *Lectures on Discrete Geometry*, volume 212. Springer Science & Business Media, 2002.

[3] Vlastimil Pták. A remark on approximation of continuous functions. *Czechoslovak Mathematical Journal*, 8(2):251–256, 1958.