Hilbert Projection Theorem

Lemma 1, (Hilbert Projection, Euclidean Case)**.** *Every nonempty, closed, and convex $K \subset \mathbb{R}^n$ contains a unique vector of minimum L_2 norm.*

Proof. See Tao, *Epsilon of Room, Vol. 1*, Proposition 1.4.12 for a proof of the general case. □

Separating Hyperplane Theorem

Theorem 1, (Separating Hyperplane Theorem)**.** *Suppose $A, B \subset \mathbb{R}^n$ are disjoint, convex, and nonempty. Then there exist $c \in \mathbb{R}$ and nonzero $v \in \mathbb{R}^n$ such that C and D lie on opposite sides (closed half-spaces) of the* ***separating hyperplane*** *$\mathcal{H} = \{x \in \mathbb{R}^n \mid v^T x = c\}$, that is, $v^T x \geq c$ and $v^T y \leq c$ for all $x \in A, y \in B$.*

(1) Consider the Minkowski sum $K \equiv A + (-B) = \{x - y \mid x \in A, y \in B\}$.
  (a) K and its closure $\overline{K}$ are convex, being the sum of two convex sets A and $-B$.
  (b) K contains the zero vector if and only if sets A and B intersect.
  (c) $\overline{K}$ may contain the zero vector even if A and B are disjoint but infinitesimally close.

(2) Reduction: It suffices to show $\langle u, v\rangle \geq 0$ for all $u \in K$ and some nonzero $v \in \mathbb{R}^n$, the **separating axis**.
  (a) Equivalently, $\langle x - y, v\rangle \geq 0$ for all $x \in A, y \in B$, by construction of K.
  (b) Then, by linearity and properties of sup and inf [1],

$$\begin{aligned} \langle x, v\rangle_2 &\geq \langle y, v\rangle_2 && \forall\, y \in B, x \in A \\ \langle x, v\rangle_2 &\geq \sup_{y\in B}\langle y, v\rangle_2 && \forall\, x \in A \\ \inf_{x\in A}\langle x, v\rangle_2 &\geq \sup_{y\in B}\langle y, v\rangle_2 \end{aligned}$$

  (c) Choose c between (or equal to) the values above to obtain a separating hyperplane.

(3) Reduction: It suffices to show $\|v\|_2 \leq \|v + t(u - v)\|_2$ for some nonzero $v \in \mathbb{R}^n$ and every $u \in K, t \in [0, 1]$.
  (a) Then, $\|v\|_2^2 \leq \|v\|_2^2 + 2t\langle v, u - v\rangle_2 + t^2\|u - v\|_2^2$.
  (b) For $0 < t \leq 1$ we thus have $0 \leq 2\langle v, u\rangle_2 - 2\|v\|_2^2 + t\|u - v\|_2^2$.
  (c) Letting $t \to 0$ gives $\langle u, v\rangle \geq \|v\|_2^2 \geq 0$ for all $u \in K$, and we may apply the previous claim.

(4) Case: Separation holds when $\operatorname{dist}(A, B) > 0$. This includes the special case where both A and B are closed and one is bounded.
  (a) Let $v \in \overline{K}$ be the unique vector in $\overline{K}$ of smallest norm given by the Hilbert projection theorem.
  (b) Because $\operatorname{dist}(A, B) > 0$, $\overline{K}$ cannot contain the origin and so v is nonzero.
  (c) Since $\overline{K}$ is convex, for any $u \in K$ the line segment $v + t(u - v)$ lies in $\overline{K}$ for any $0 \leq t \leq 1$.
  (d) By minimality of v, $\|v\|_2 \leq \|v + t(u - v)\|_2$. Using the second reduction above, we are done.

(5) Case: Separation holds when $\operatorname{dist}(A, B) = 0$ and the interior K° is nonempty.
  (a) Then, the interior can be written as a union of countably many nonempty, compact, convex subsets, $K^\circ = \bigcup_{j=1}^{\infty} K_k$. For example, $K_j = \overline{(1 - \frac{1}{j})K} \cap \overline{B(0, j)}$.
  (b) Let $v_j \in K_j$ be the unique vector of smallest norm in K_j given by the projection theorem.
    (i) Since $0 \notin K^\circ$, we also have $0 \notin K_j$, so each v_j is nonzero.
    (ii) By an argument similar to the previous case, $\langle u, v_j\rangle \geq 0$ for all $u \in K_j$.
  (c) Normalize the v_j to have unit length. By compactness of the unit sphere, the sequence $(v_j)_{j=1}^{\infty}$ has a subsequential limit $v \in \mathbb{R}^n$, which is nonzero.
  (d) By continuity of inner products, $\langle u, v\rangle \geq 0$ for all $u \in K$, and we are done.

(6) Case: Finally, if K has empty interior, then K is entirely contained by some hyperplane $\langle \cdot, v\rangle = c$, which we may used for (weak) separation.

Separating Axis Theorem

Theorem 2, (Separating Axis Theorem, 2D)**.** *Suppose $A, B \subset \mathbb{R}^2$ are disjoint, convex, compact polygons. Then there exists a separating line with normal vector orthogonal to one of the edges of the Minkowski sum $A + (-B)$.*

Proof. Adapted from an answer on Math StackExchange, see https://math.stackexchange.com/q/2106402.

(1) Choosing the Axis. The Minkowski sum $K = A + (-B)$ is also a compact, convex polygon, so we can express $K = \bigcap_{k=1}^{n} \mathcal{H}_k$ as the intersection of finitely many closed half-planes $\mathcal{H}_k \subset \mathbb{R}^n$. Since $A \cap B = \emptyset$, we have $0 \notin K$, and accordingly $0 \notin \mathcal{H}_k$ for some k. Therefore, the vector $v \in \mathcal{H}_k$ of smallest norm given by the projection theorem is nonzero.

(2) Orthogonality. Denote by $\ell_k \subset \mathcal{H}_k$ the line corresponding to half-plane $\mathcal{H}_k$. Let $w \in \mathbb{R}^n$ be a unit vector in the direction of ℓ_k. Then, $v - \alpha w \in \ell_k$ for all $\alpha \in \mathbb{R}$, and by minimality of v,

$$\|v\|_2^2 \leq \|v - \alpha w\|_2^2 = \|v\|_2^2 - 2\alpha \langle v, w \rangle_2 + \alpha^2$$

Choosing $\alpha = \langle v, w \rangle$, we find that $\|v\|_2^2 \leq \|v\|_2^2 - \langle v, w \rangle_2^2$, hence $\langle v, w \rangle = 0$ and $v \perp \ell_k$.

(3) Separation. From the proof of the hyperplane separation theorem, it suffices to show $\|v\|_2^2 \leq \|v + t(x - v)\|_2^2$ for all $x \in K$ and $0 \leq t \leq 1$. Recall $K \subset \mathcal{H}_k$, so by convexity, $v + t(x - v) \in \mathcal{H}_k$. By minimality of v, the desired inequality holds and we are done! □

Corollary 1. *Suppose $A, B \subset \mathbb{R}^2$ are disjoint, convex, compact polygons. Then there exists a separating line with normal vector orthogonal to one of the edges of A or B.*

Theorem 3, (Separating Axis Theorem, General Case)**.** *Suppose $A, B \subset \mathbb{R}^n$ are disjoint, convex, compact polytopes. Then there exists a separating hyperplane with normal vector orthogonal to one of the facets of the Minkowski sum $A + (-B)$.*

References

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[2] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[3] Jiří Matoušek. *Lectures on Discrete Geometry*, volume 212. Springer Science & Business Media, 2002.

[4] Ralph Tyrell Rockafellar. *Convex Analysis*. Princeton University Press, 1970.