Property 1. $$\begin{aligned}(I+P)^{-1} &= (I+P)^{-1}(I+P-P) \\ &= I-(I+P)^{-1}P\end{aligned}$$

Property 2. $$\begin{aligned}P+PQP = P(I+QP) &= (I+PQ)P \\ (I+PQ)^{-1}P &= P(I+QP)^{-1}\end{aligned}$$

Lemma 1, (Matrix Inversion, v1). *For invertible A but general (rectangular) B, C, and D,*

$$(A+BCD)^{-1} = A^{-1} - A^{-1}BCDA^{-1}(I+BCDA^{-1})^{-1}$$

Proof. Using Property 1,

$$\begin{aligned}(A+BCD)^{-1} &= \Big[A(I+A^{-1}BCD)\Big]^{-1} \\ &= \Big[I+A^{-1}BCD\Big]^{-1}A^{-1} \\ &= \Big[I-(I+A^{-1}BCD)^{-1}A^{-1}BCD\Big]A^{-1} && \text{(Property 1)} \\ &= A^{-1}-(I+A^{-1}BCD)^{-1}A^{-1}BCDA^{-1}\end{aligned}$$

Repeatedly applying Property 2 produces

$$\begin{aligned}(A+BCD)^{-1} &= A^{-1}-(I+A^{-1}BCD)^{-1}A^{-1}BCDA^{-1} \\ &= A^{-1}-A^{-1}(I+BCDA^{-1})^{-1}BCDA^{-1} \\ &= A^{-1}-A^{-1}B(I+CDA^{-1}B)^{-1}CDA^{-1} \qquad (1) \\ &= A^{-1}-A^{-1}BC(I+DA^{-1}BC)^{-1}DA^{-1} \\ &= A^{-1}-A^{-1}BCD(I+A^{-1}BCD)^{-1}A^{-1} \\ &= A^{-1}-A^{-1}BCDA^{-1}(I+BCDA^{-1})^{-1}\end{aligned}$$ □

Lemma 2, (Matrix Inversion, v2). *For invertible A and C but general (rectangular) B and D,*

$$\begin{aligned}(A+BCD)^{-1} &= A^{-1}-A^{-1}B(I+CDA^{-1}B)^{-1}CDA^{-1} && \text{(eqn. 1)} \\ &= A^{-1}-A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}\end{aligned}$$

Lemma 3, (Matrix Inversion, v3). *A different use of Property 2 gives*

$$\begin{aligned}(A+BCD)^{-1}BC &= \big[(I+BCDA^{-1})A\big]^{-1}BC \\ &= A^{-1}(I+BCDA^{-1})^{-1}BC \\ &= A^{-1}B(I+CDA^{-1}B)^{-1}C && \text{(Property 2)} \\ &= A^{-1}B(C^{-1}+DA^{-1}B)^{-1} && \text{(for invertible } C\text{)}\end{aligned}$$