Constructing probability measures for *all* sets in a general σ-algebra would be near impossible. The Carathéodory extension theorem allows us to define a measure explicitly for only a small collection of simple sets, which may or may not form a σ-algebra, and automatically extend the measure to a proper measurable space. The uniqueness claim in the extension theorem makes use of **Dynkin's π-λ theorem**. The name refers to π- and λ-systems, which are convenient names attached two the two sets of technical conditions that appear in Dynkin's theorem. A π-system is a class of subsets closed under finite intersection, while a λ-system satisfies slightly weaker conditions than a σ-algebra.

Definition 1. A class $\mathcal{C} \subset \mathcal{P}(\Omega)$ is a **π-system** if $A, B \in \mathcal{C} \implies A \cap B \in \mathcal{C}$.

Definition 2. A class $\mathcal{L} \subset \mathcal{P}(\Omega)$ is a **λ-system** if

I. $\Omega \in \mathcal{L}$
II. (Contained Difference) $A, B \in \mathcal{L}, A \subset B \implies B \setminus A \in \mathcal{L}$
III. (Monotone Union) $A_1 \subset A_2 \subset \cdots \in \mathcal{L} \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{L}$

Exercise 1. Prove each of the following statements.

I. Every σ-algebra is both a π-system and λ-system.
II. Every algebra is a π-system, but an algebra need not be a λ-system.
III. A λ-system that is also a π-system is a σ-algebra.

It is easily checked that the intersection of two λ-systems is a λ-system. Much like for σ-algebras, we define the λ-system $\langle \mathcal{A} \rangle_\lambda$ generated by $\mathcal{A} \subset \mathcal{P}(\Omega)$ to be the intersection of all λ-systems containing $\mathcal{A}$. It is the smallest λ-system containing $\mathcal{A}$.

Lemma 1. *A class $\mathcal{A} \subset \mathcal{P}(\Omega)$ of subsets of Ω which is both a π-system and λ-system is a σ-algebra.*

Lemma 2. *If $\mathcal{C} \subset \mathcal{P}(\Omega)$ is a π-system, then $\langle \mathcal{C} \rangle_\lambda$ is a π-system.*

(1) First, we show $\langle \mathcal{C} \rangle_\lambda$ is closed under intersection with sets from $\mathcal{C}$.
- (a) Observe that the set $\Lambda_1 = \{A \in \langle \mathcal{C} \rangle_\lambda \mid \forall B \in \mathcal{C}, A \cap B \in \langle \mathcal{C} \rangle_\lambda\}$ is a λ-system containing $\mathcal{C}$.
  - (i) $\Omega \in \Lambda_1$, since for any $B \in \mathcal{C} \subset \langle \mathcal{C} \rangle_\lambda$, we have $\Omega \cap B = B \in \langle \mathcal{C} \rangle_\lambda$.
  - (ii) Contained Difference. Suppose $A_1, A_2 \in \Lambda_1$, with $A_1 \subset A_2$.
    - (1) For any $B \in \mathcal{C}$, we know $A_1 \cap B, A_2 \cap B \in \langle \mathcal{C} \rangle_\lambda$.
    - (2) Recalling that $\langle \mathcal{C} \rangle_\lambda$ is a λ-system, and therefore closed under contained differences,

$$(A_2 \setminus A_1) \cup B = (A_2 \cup B) \setminus (A_1 \cup B) \in \langle \mathcal{C} \rangle_\lambda$$

    - (3) Therefore $A_2 \setminus A_1 \in \Lambda_1$, and Λ_1 is closed under contained differences.
  - (iii) Monotone Union. Suppose $A_1 \subset A_2 \subset \cdots \in \Lambda_1$. Let $A = \bigcup_{n=1}^{\infty} A_n$.
    - (1) Pick any $B \in \mathcal{C}$. Then $B_n \equiv A_n \cap B \in \langle \mathcal{C} \rangle_\lambda$, and $B_1 \subset B_2 \subset \cdots$.
    - (2) Recalling that $\langle \mathcal{C} \rangle_\lambda$ is a λ-system, and therefore closed under monotone unions,

$$A \cap B = \bigcup_{n=1}^{\infty} [A_n \cap B] = \bigcup_{n=1}^{\infty} B_n \in \langle \mathcal{C} \rangle_\lambda$$

    - (3) Therefore, Λ_1 is closed under monotone unions.
- (b) Therefore, $\langle \mathcal{C} \rangle_\lambda \subset \Lambda_1$ by definition. But $\Lambda_1 \subset \langle \mathcal{C} \rangle_\lambda$, so the two sets are equal!

(2) Next, we use the previous result to show $\langle \mathcal{C} \rangle_\lambda$ is closed under intersection in general.
- (a) Observe that the set $\Lambda_2 = \{A \in \langle \mathcal{C} \rangle_\lambda \mid \forall B \in \langle \mathcal{C} \rangle_\lambda, A \cap B \in \langle \mathcal{C} \rangle_\lambda\}$ is a λ system.
- (b) By the previous result, $\mathcal{C} \subset \Lambda_2$, as the intersection of any two sets from $\langle \mathcal{C} \rangle_\lambda$ and $\mathcal{C}$ lies in $\langle \mathcal{C} \rangle_\lambda$.
- (c) Therefore, $\langle \mathcal{C} \rangle_\lambda \subset \Lambda_2$, by definition. But $\Lambda_2 \subset \langle \mathcal{C} \rangle_\lambda$, so the two sets are equal!

Theorem 1, (Dynkin π-λ)**.** *If $\mathcal{C} \subset \mathcal{P}(\Omega)$ is a π-system, then $\langle \mathcal{C} \rangle_\lambda = \langle \mathcal{C} \rangle_\sigma$.*

Proof. We already know $\langle \mathcal{C} \rangle_\lambda$ is a λ-system. Applying Lemma 2, $\langle \mathcal{C} \rangle_\lambda$ is also a π-system. By Lemma 1, then, $\langle \mathcal{C} \rangle_\lambda$ is a σ-algebra containing $\mathcal{C}$, and so $\langle \mathcal{C} \rangle_\sigma \subset \langle \mathcal{C} \rangle_\lambda$. Similarly, $\langle \mathcal{C} \rangle_\lambda \subset \langle \mathcal{C} \rangle_\sigma$, since every σ-algebra is a λ-system. Therefore, $\langle \mathcal{C} \rangle_\lambda = \langle \langle C \rangle_\lambda \rangle_\sigma$. □

Corollary 1. *If $\mathcal{C} \subset \mathcal{P}(\Omega)$ is a π-system and $\mathcal{L} \subset \mathcal{P}(\Omega)$ is a λ-system containing $\mathcal{C}$, then $\langle \mathcal{C} \rangle_\sigma \subset \mathcal{L}$.*

Proof. By the π-λ theorem, $\langle \mathcal{C} \rangle_\sigma = \langle \mathcal{C} \rangle_\lambda \subset \mathcal{L}$. □